

CORRELATION FUNCTIONS OF REAL ZEROS OF RANDOM POLYNOMIALS

FRIEDRICH GÖTZE, DZIANIS KALIADA, AND DMITRY ZAPOROZHETS

ABSTRACT. We give an explicit formula for the correlation functions of real zeros of a random polynomial with arbitrary independent continuously distributed coefficients.

1. INTRODUCTION

Let $\xi_0, \xi_1, \dots, \xi_n$ be independent random variables with probability density functions f_0, \dots, f_n . Consider a random polynomial

$$G(x) = \xi_n x^n + \xi_{n-1} x^{n-1} + \dots + \xi_1 x + \xi_0, \quad x \in \mathbb{R}^1.$$

With probability one, all zeros of G are simple. Denote by μ the empirical measure counting the real zeros of G :

$$\mu = \sum_{x: G(x)=0} \delta_x,$$

where δ_x is the unit point mass at x . The distribution of μ can be described by its *correlation functions* (also known as *joint intensities*). Recall (see, e.g., [6]) that the correlation functions of μ are functions (if well-defined) $\rho_k : \mathbb{R}^k \rightarrow \mathbb{R}^+$ for $k = 1, \dots, n$, such that for any family of mutually disjoint Borel subsets $B_1, \dots, B_k \subset \mathbb{R}^1$,

$$\mathbb{E} \left[\prod_{i=1}^k \mu(B_i) \right] = \int_{B_1} \dots \int_{B_k} \rho_k(x_1, \dots, x_k) dx_1 \dots dx_k.$$

A standard tool for evaluating ρ_k is the following extension of the Kac-Rice formula (see [1],[2]):

$$(1) \quad \rho_k(x_1, \dots, x_k) = \int_{\mathbb{R}^k} |t_1 \dots t_k| D_k(\mathbf{0}, \mathbf{t}, x_1, \dots, x_k) dt_1 \dots dt_k,$$

where $\mathbf{t} = (t_1, \dots, t_k)$ and $D_k(\cdot, \cdot, x_1, \dots, x_k)$ is the joint probability density function of the random vectors

$$(G(x_1), \dots, G(x_k)) \quad \text{and} \quad (G'(x_1), \dots, G'(x_k)).$$

The goal of this paper is to provide more explicit expressions for $\rho_k(\mathbf{x})$. The main tool that we will use is the *Coarea formula* (see Lemma 5.2).

Our methods can be applied to the case of dependent coefficients having arbitrary joint probability density function. For simplicity, we consider only the case of independent coefficients.

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2. MAIN RESULT

Let us start with some notation. Denote

$$\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k.$$

We use the following notation for the elementary symmetric polynomials:

$$\sigma_i(\mathbf{x}) := \begin{cases} \sum_{1 \leq j_1 < \dots < j_i \leq k} x_{j_1} x_{j_2} \dots x_{j_i}, & \text{if } 0 \leq i \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $V(\mathbf{x})$ the Vandermonde matrix

$$V(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & \dots & x_1^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \dots & x_k^{k-1} \end{pmatrix}.$$

To formulate the first result, consider the random function $\eta = (\eta_0, \dots, \eta_{k-1})^T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ defined as

$$(2) \quad \eta(\mathbf{x}) = -V^{-1}(\mathbf{x}) \begin{pmatrix} \sum_{j=k}^n \xi_j x_1^j \\ \vdots \\ \sum_{j=k}^n \xi_j x_k^j \end{pmatrix}.$$

Theorem 2.1. *We have*

$$(3) \quad \rho_k(\mathbf{x}) = \prod_{1 \leq i < j \leq k} |x_i - x_j|^{-1} \times \mathbb{E} \left[\prod_{i=1}^k \left| \sum_{j=0}^{k-1} j \eta_j(\mathbf{x}) x_i^{j-1} + \sum_{j=k}^n j \xi_j x_i^{j-1} \right| \prod_{i=0}^{k-1} f_i(\eta_i(\mathbf{x})) \right].$$

Theorem 2.1 has been stated in [11] without detailed proof.

It is possible to obtain an explicit expression for $\eta(\mathbf{x})$ in terms of the *Schur functions*. Recall that for a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of length $\leq k$ the Schur function $S_\lambda(\mathbf{x})$ is given by

$$S_\lambda(\mathbf{x}) = \frac{\det(x_i^{\lambda_{k-j+1}+j-1})_{1 \leq i, j \leq k}}{\prod_{1 \leq i < j \leq k} (x_j - x_i)}.$$

Proposition 2.2. *For $i = 1, \dots, k$, we have*

$$\eta_i(\mathbf{x}) = (-1)^{k-i} \sum_{j=k}^n \xi_j S_{\lambda_{ij}}(\mathbf{x}),$$

where the partition λ_{ij} is defined as

$$\lambda_{ij} = (j - k + 1, \underbrace{1, \dots, 1}_{k-i-1}, \underbrace{0, \dots, 0}_i).$$

For the basic properties of the Schur functions see, e.g., [9, Section 1.3].

Now we are ready to state our main result.

Theorem 2.3. *We have*

$$\rho_k(\mathbf{x}) = \prod_{1 \leq i < j \leq k} |x_i - x_j| \times \int_{\mathbb{R}^{n-k+1}} \prod_{i=0}^n f_i \left(\sum_{j=0}^{n-k} (-1)^{k-i+j} \sigma_{k-i+j}(\mathbf{x}) t_j \right) \prod_{i=1}^k \left| \sum_{j=0}^{n-k} t_j x_i^j \right| dt_0 \dots dt_{n-k}.$$

Corollary 2.4. *For $k = n$ we have*

$$(4) \quad \rho_n(\mathbf{x}) = \prod_{1 \leq i < j \leq n} |x_i - x_j| \int_{-\infty}^{\infty} |t|^n \prod_{i=0}^n f_i((-1)^{n-i} \sigma_{n-i}(\mathbf{x})t) dt.$$

3. UNIFORMLY DISTRIBUTED COEFFICIENTS

In algebraic number theory, random polynomials with independent and uniformly distributed on $[-1, 1]$ coefficients are of special interest (see [7],[5],[4]). Let us apply Theorem 2.3 to this case.

Suppose that

$$f_i = \frac{1}{2} \mathbb{1}[-1, 1], \quad i = 0, \dots, n.$$

Then it follows from Theorem 2.3 that

$$\rho_k(\mathbf{x}) = 2^{-n-1} \prod_{1 \leq i < j \leq k} |x_i - x_j| \int_{D_{\mathbf{x}}} \prod_{i=1}^k \left| \sum_{j=0}^{n-k} t_j x_i^j \right| dt_0 \dots dt_{n-k},$$

where the domain of integration $D_{\mathbf{x}}$ is defined as

$$D_{\mathbf{x}} = \left\{ (t_0, \dots, t_{n-k}) \in \mathbb{R}^{n-k+1} : \max_{0 \leq i \leq n} \left| \sum_{j=0}^{n-k} (-1)^{i-j} \sigma_{i-j}(\mathbf{x}) t_j \right| \leq 1 \right\}.$$

In particular,

$$\rho_n(\mathbf{x}) = \frac{2^{-n}}{n+1} \cdot \frac{\prod_{1 \leq i < j \leq n} |x_i - x_j|}{(\max_{0 \leq i \leq n} |\sigma_i(\mathbf{x})|)^{n+1}}.$$

4. THE n -POINT CORRELATION FUNCTION

It follows from the properties of the correlation functions (see, e.g., [6]) that

$$\mathbb{E} [\mu(\mathbb{R}^1)(\mu(\mathbb{R}^1) - 1) \dots (\mu(\mathbb{R}^1) - n + 1)] = \int_{\mathbb{R}^n} \rho_n(\mathbf{x}) d\mathbf{x}.$$

Since $\mu(\mathbb{R}^1) \leq n$, we can calculate the probability that all zeros of G are real:

$$\begin{aligned} \mathbb{P}(\mu(\mathbb{R}^1) = n) &= \frac{1}{n!} \int_{\mathbb{R}^n} \rho_n(\mathbf{x}) d\mathbf{x} \\ &= \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} |x_i - x_j| \int_{-\infty}^{\infty} |t|^n \prod_{i=0}^n f_i((-1)^{n-i} \sigma_{n-i}(\mathbf{x})t) dt d\mathbf{x}. \end{aligned}$$

This formula has been obtained earlier in [10].

Let us calculate ρ_n for some specific distributions.

4.1. Gaussian distribution. Suppose that

$$f_i(t) = \frac{1}{\sqrt{2\pi}v_i} \exp\left(-\frac{t^2}{2v_i^2}\right), \quad i = 0, \dots, n.$$

Using the formula for the n -th absolute moment of the Gaussian distribution, it follows from (4) that

$$\rho_n(\mathbf{x}) = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{(2\pi)^{n/2}v_0 \dots v_n} \left(\sum_{i=0}^n \frac{\sigma_{n-i}^2(\mathbf{x})}{v_i^2} \right)^{-(n+1)/2} \prod_{1 \leq i < j \leq n} |x_i - x_j|.$$

In particular, for $v_i = v_j$ we have

$$\rho_n(\mathbf{x}) = \frac{\sqrt{2}\Gamma\left(\frac{n+1}{2}\right)}{(2\pi)^{n/2}} \left(\sum_{i=0}^n \sigma_{n-i}^2(\mathbf{x}) \right)^{-(n+1)/2} \prod_{1 \leq i < j \leq n} |x_i - x_j|.$$

4.2. Exponential distribution. Suppose that

$$f_i(t) = \exp(-t) \mathbb{1}\{t \geq 0\}, \quad i = 0, \dots, n.$$

Then with probability one G does not have positive real zeros. Hence $\rho_n(\mathbf{x}) > 0$ only if \mathbf{x} lies in the negative orthant \mathbb{R}_-^k . In this case we have $(-1)^i \sigma_i(\mathbf{x}) \geq 0$ and by some elementary transformations, (4) implies

$$\rho_n(\mathbf{x}) = n! \left(\sum_{i=0}^n (-1)^i \sigma_i(\mathbf{x}) \right)^{-(n+1)} \prod_{1 \leq i < j \leq n} |x_i - x_j| \mathbb{1}\{\mathbf{x} \in \mathbb{R}_-^k\}.$$

Using the well-known identity

$$\sum_{i=0}^n (-1)^i \sigma_i(\mathbf{x}) = \prod_{i=1}^n (1 - x_i),$$

we obtain

$$\rho_n(\mathbf{x}) = n! \frac{\prod_{1 \leq i < j \leq n} |x_i - x_j|}{\prod_{i=1}^n (1 - x_i)^{n+1}} \mathbb{1}\{\mathbf{x} \in \mathbb{R}_-^k\}.$$

5. PROOF OF THEOREM 2.1

Obviously, $G(x_1) = \dots = G(x_k) = 0$ if and only if

$$(5) \quad \begin{pmatrix} 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_k & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \end{pmatrix} = \mathbf{0},$$

which is equivalent to

$$\begin{pmatrix} x_1^k & x_1^{k+1} & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_k^k & x_k^{k+1} & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_k \\ \vdots \\ \xi_n \end{pmatrix} = -V(\mathbf{x}) \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{k-1} \end{pmatrix}.$$

Recalling (2), we obtain that (5) is equivalent to

$$\eta(\mathbf{x}) = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{k-1} \end{pmatrix}.$$

Denote by $J_\eta(\mathbf{x})$ the Jacobian matrix of η at the point \mathbf{x} .

Lemma 5.1.

$$\det J_\eta(\mathbf{x}) = - \frac{\prod_{i=1}^k \left(\sum_{j=0}^{k-1} j \eta_j(\mathbf{x}) x_i^{j-1} + \sum_{j=k}^n j \xi_j x_i^{j-1} \right)}{\prod_{1 \leq i < j \leq k} (x_j - x_i)}.$$

Proof. Differentiating

$$V(\mathbf{x})\eta(\mathbf{x}) = - \begin{pmatrix} x_1^k & x_1^{k+1} & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_k^k & x_k^{k+1} & \dots & x_k^n \end{pmatrix} \begin{pmatrix} \xi_k \\ \vdots \\ \xi_n \end{pmatrix},$$

we obtain

$$\begin{aligned} V(\mathbf{x})J_\eta(\mathbf{x}) + \text{diag} \left(\sum_{j=0}^{k-1} j \eta_j(\mathbf{x}) x_1^{j-1}, \dots, \sum_{j=0}^{k-1} j \eta_j(\mathbf{x}) x_k^{j-1} \right) \\ = - \text{diag} \left(\sum_{j=k}^n j \xi_j x_1^{j-1}, \dots, \sum_{j=k}^n j \xi_j x_k^{j-1} \right). \end{aligned}$$

We finish the proof by taking the second term from the left hand side to the right hand side and using

$$\det V(\mathbf{x}) = \prod_{1 \leq i < j \leq k} (x_j - x_i).$$

□

Lemma 5.2 (Coarea formula). *Let $B \subset \mathbb{R}^k$ be a region. Let $u : B \rightarrow \mathbb{R}^k$ be a Lipschitz function and $h : \mathbb{R}^k \rightarrow \mathbb{R}^1$ be an L^1 -function. Then*

$$\int_{\mathbb{R}^k} \# \{ \mathbf{x} \in B : u(\mathbf{x}) = \mathbf{y} \} h(\mathbf{y}) d\mathbf{y} = \int_B |\det J_u(\mathbf{x})| h(u(\mathbf{x})) d\mathbf{x},$$

where $J_u(\mathbf{x})$ is the Jacobian matrix of $u(\mathbf{x})$.

Proof. See [3, pp. 243–244].

□

Let B_1, \dots, B_k be a family of mutually disjoint Borel subsets in \mathbb{R}^1 . Denote $B = B_1 \times \dots \times B_k \subset \mathbb{R}^k$. We have

$$\begin{aligned} \mathbb{E} \left[\prod_{i=1}^k \mu(B_i) \right] &= \mathbb{E} \# \{ \mathbf{x} \in B : \eta(\mathbf{x}) = (\xi_0, \dots, \xi_{k-1}) \} \\ &= \mathbb{E} \int_{\mathbb{R}^k} \# \{ \mathbf{x} \in B : \eta(\mathbf{x}) = \mathbf{y} \} f_0(y_0) \dots f_{k-1}(y_{k-1}) d\mathbf{y}. \end{aligned}$$

Applying Lemma 5.2 and Fubini's theorem to the right hand side, we obtain

$$\mathbb{E} \left[\prod_{i=1}^k \mu(B_i) \right] = \int_B \mathbb{E} |\det J_\eta(\mathbf{x})| f_0(\eta_0(\mathbf{x})) \dots f_{k-1}(\eta_{k-1}(\mathbf{x})) d\mathbf{x}.$$

Now the proof of Theorem 2.1 follows from Lemma 5.1.

6. PROOF OF THEOREM 2.3

Theorem 2.1 states that

$$\begin{aligned} (6) \quad \rho_k(\mathbf{x}) &= \prod_{1 \leq i < j \leq k} |x_j - x_i|^{-1} \\ &\times \int_{\mathbb{R}^{n-k+1}} \prod_{i=1}^k \left| \sum_{j=0}^n j a_j x_i^{j-1} \right| \prod_{i=0}^n f_i(a_i) da_k da_{k+1} \dots da_n, \end{aligned}$$

where a_0, \dots, a_{k-1} are functions of a_k, \dots, a_n and x_1, \dots, x_k :

$$(7) \quad \begin{pmatrix} a_0 \\ \vdots \\ a_{k-1} \end{pmatrix} = -V^{-1}(\mathbf{x}) \begin{pmatrix} \sum_{j=k}^n a_j x_1^j \\ \vdots \\ \sum_{j=k}^n a_j x_k^j \end{pmatrix}.$$

To prove the theorem, we will use the ideas from [8, pp. 58–59] and [7, Lemmas 5 and 6]. Equation (7) means that x_1, x_2, \dots, x_k are zeros of the polynomial $\sum_{j=0}^n a_j x^j$. Hence there exists a unique polynomial $\sum_{j=0}^{n-k} b_j x^j$ such that

$$(8) \quad \sum_{j=0}^n a_j x^j = \prod_{i=1}^k (x - x_i) \left(\sum_{j=0}^{n-k} b_j x^j \right) = \left(\sum_{j=0}^k (-1)^{k-j} \sigma_{k-j}(\mathbf{x}) x^j \right) \left(\sum_{j=0}^{n-k} b_j x^j \right).$$

The variables a_0, \dots, a_n are uniquely defined by \mathbf{x} and b_0, \dots, b_{n-k} from (8):

$$(9) \quad a_i = \sum_{j=0}^{n-k} (-1)^{k-i+j} \sigma_{k-i+j}(\mathbf{x}) b_j.$$

Thus we can change variables in (6) substituting a_k, \dots, a_n by their expressions in terms of \mathbf{x} and b_0, \dots, b_{n-k} from (9). The Jacobian of this substitution is a lower triangle matrix with unities in the diagonal. Hence its determinant is equal to one.

Differentiating (8) at the point x_i we get

$$(10) \quad \sum_{j=0}^n j a_j x_i^{j-1} = \prod_{j \neq i} (x_i - x_j) (b_{n-k} x_i^{n-k} + \dots + b_1 x_i + b_0), \quad i = 1, \dots, k.$$

Substituting (9) and (10) in (6) finishes the proof.

7. PROOF OF PROPOSITION 2.2

For $0 \leq i \leq k-1$ and $j \geq k$, denote by $V_{ij}^*(\mathbf{x})$ the matrix obtained from $V(\mathbf{x})$ by substitution of the i -th column by $(x_1^j, \dots, x_k^j)^T$:

$$V_{ij}^*(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & \dots & x_1^{r-1} & x_1^j & x_1^{r+1} & \dots & x_1^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & \dots & x_k^{r-1} & x_k^j & x_k^{r+1} & \dots & x_k^{k-1} \end{pmatrix}.$$

Then by Cramer's rule, we have

$$\eta_r(\mathbf{x}) = -\frac{1}{\det V(\mathbf{x})} \sum_{j=k}^n \xi_j \det V_{ij}^*(\mathbf{x}).$$

It is easily seen that

$$\frac{V_{ij}^*(\mathbf{x})}{\det V(\mathbf{x})} = (-1)^{k-i-1} S_{\lambda_{ij}}(\mathbf{x}),$$

and the proof follows.

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FRIEDRICH GÖTZE, FACULTY OF MATHEMATICS, BIELEFELD UNIVERSITY, P. O. Box 10 01 31,
33501 BIELEFELD, GERMANY

E-mail address: `goetze@math.uni-bielefeld.de`

DZIANIS KALIADA, INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF BE-
LARUS, 220072 MINSK, BELARUS

E-mail address: `koledad@rambler.ru`

DMITRY ZAPOROZHETS, ST. PETERSBURG DEPARTMENT OF STEKLOV INSTITUTE OF MATHEMAT-
ICS, FONTANKA 27, 191011 ST. PETERSBURG, RUSSIA

E-mail address: `zap1979@gmail.com`